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Minimally Coupled FRW Cosmologies as Dynamical Systems

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The dynamical evolution of FRW cosmologies, minimally coupled to a finite set of scalar fields with an arbitrary potential, is studied. The general properties of the scale factor and the scalar fields, which are independent of the potential, are determined. It is shown that for $k = 0, -1$ the evolution of the Hubble function is growing, independently of the potential, which allows expansive and contractive evolutions of the scale factor. Moreover, if the potential can take negative values, cyclic universes are possible. In the spherical geometry case, $k = 1$, the existence of expansive, contractive, or cyclic, universes is possible, independently of the condition stated above, namely that the potential would necessarily take negative values. Moreover, the existence of chaotic solutions can be obtained via a fine-tuning.

KEY WORDS: cosmology; dynamical systems; astrophysics.

1. INTRODUCTION

This paper has a twofold motivation:

1. We continue our research about the cosmological models as dynamical systems (see Castagnino *et al.*, 2000, 2001, 2002), following the same philosophy. Namely even if the numerical methods and experiments are powerful tools for the comprehension of these systems, the obtained results are only clues, not rigorous proofs, of the general properties that we would like to attribute to these systems. So we have followed a different route: using the theory of dynamical systems to search the mathematical properties of each model, in such a way to prepare the way towards the physical comprehension of these systems.

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2. To present to cosmologists a detailed study of FRW-cosmologies as dynamical systems. Cornish and Lewin (1996), Easther and Maeda (1999), Charters (2001), Peebles (1993), and many others several authors have studied the dynamical behavior of FRW cosmologies conformally or minimally coupled to a scalar field. Even before the nineties, for several reasons the current models usually contains one (or many) scalar field with an arbitrary potential $V(\psi)$, e.g. power law potential, as in Linde (1983) or Peebles and Vilenkin (1999a), Liddle and Scherrer (1999b), Kolda and Lyth (1999c), inverse power-law potential, as in Ratra and Peebles (1988) and Peebles and Ratra (1988), exponential potential, as in Ratra and Peebles (1988) and Peebles and Ratra (1988), etc. Since the arrival of $\Lambda > 0$ in the late nineties, even more than before, models with arbitrary $V(\psi)$ appears very frequently in the literature. For example a very interesting study with negative potential can be found in Felder *et al.* (2002) and we will try to continue this line considering more general models.

Our final aim would be to offer the most general possible formalism with arbitrary potential, many scalar fields, and $k = 0, \pm 1$, with the most complete account of the dynamical properties of these cosmological models considered as dynamical systems, e.g. the conditions for the existence of cyclic universes (Steinhardt and Turok, 2002a,b) or the oscillatory behavior of the scale factor.

The paper is organized as follows: in Section 2 we present the FRW model with n scalar fields minimally coupled, with a generic potential. In Section 3, we find the fixed point of the corresponding dynamics. In Section 4, we find the Lyapunov function for positive definite potentials. In Section 5, we study the evolution of the scalar fields and their energy density. In Section 6, we present, a very simple way to classify the different solution introducing the HVZ space of the model. In Section 7, we find and study some particular oscillatory solutions. In Section 8, we draw our conclusions.

2. COSMOLOGY WITH *n* SCALAR FIELD

In this work we study a spatially closed, flat, or open FRW minimally coupled to n neutral scalar fields. The metric is given by:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right), \tag{1}$$

where t is the proper time (also known as cosmic time), $k = 1, 0, -1$, and $d\Omega^2$ is the element of volume of the unit spatial sphere (either plane or hyperboloid). The Lagrangian density of the system is

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M\,, \tag{2}$$

where $\mathcal{L}_G = -\frac{1}{12} R$ is the gravitational Lagrangian density (R is the Ricci scalar) and

$$\mathcal{L}_M = \sum_{i=1}^{n} \left(-\frac{1}{2}\, \partial_\mu \psi_i \; \partial^\mu \psi_i + \frac{1}{12} R \; \psi_i^2 \right) + V(\psi_1, \ldots, \psi_n) \tag{3}$$

is the Lagrangian matter density. The Ricci scalar is related to the scale factor a by the equation

$$\frac{R}{6} = \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2}, \tag{4}$$

We have chosen $\frac{4\pi G}{3} = c = 1$ and the overdot indicates the derivative with respect to proper time t.
The evolution equations for the fields ψ_i are the Klein Gordon equations

$$\ddot{\psi}_i + 3 \frac{\dot{a}}{a} \dot{\psi}_i + \partial_{\psi_i} V(\psi_1, \ldots, \psi_n) = 0, \; i = 1, \ldots, n, \tag{5}$$

the evolution equation for the scale factor a is given by

$$\ddot{a} + 2a \left(\sum_{i=1}^{n} \dot{\psi}_i^2 - V(\psi_1, \ldots, \psi_n) \right) = 0, \tag{6}$$

and the Einstein condition reads

$$-\left(\frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} + \sum_{i=1}^{n} \dot{\psi}_i^2 + 2\, V(\psi_1, \ldots, \psi_n) = 0. \tag{7}$$

The energy density ρ and the pressure p associated with the scalar fields are

$$\rho = \frac{1}{2} \sum_{i=1}^{n} \dot{\psi}_i^2 + V(\psi_1, \ldots, \psi_n),$$

$$p = \frac{1}{2} \sum_{i=1}^{n} \dot{\psi}_i^2 - V(\psi_1, \ldots, \psi_n). \tag{8}$$

and the state equations is $w = p \,/\, \rho$.
Let us introduce the variable change $u = a^2$. Then

$$H = \frac{\dot{a}}{a} = \frac{1}{2} \frac{\dot{u}}{u}, \tag{9}$$

where H is the Hubble function. Therefore Eqs. (5) reads

$$\ddot{\psi}_i + 3H\ \dot{\psi}_i + \partial_{\psi_i} V(\psi_1, \ldots, \psi_n) = 0, \quad i = 1, \ldots, n. \tag{10}$$

and Eq. (6) becomes

$$\dot{H} = \frac{k}{u} - 3\sum_{i=1}^{n} \dot{\psi}_i^2, \tag{11}$$

and the Einstein constraint is rewritten as

$$0 = -H^2 - \frac{k}{u} + 2\,V + \sum_{i=1}^{n} \dot{\psi}_i^2\ . \tag{12}$$

Then let us state a general property: from Eq. (11), when $k = 0$ or $k = -1$, $\dot{H} < 0$ and therefore it does not change sing and H has monotonous behavior and therefore no oscillations in the scale factor, for any kind of potential and any arbitrary number of scalar fields.

Integrating Eq. (11) we have

$$H(t) = H_0 - 3\,f(t) + k\ g(t), \tag{13}$$

where H_0 is Hubble function at the initial time $t = 0$, and the function f, g are

$$f(t) = \sum_{i=1}^{n} \int_0^t \dot{\psi}_i^2\ (t')\ dt' \geqq 0,$$

$$g(t) = \int_0^t \frac{1}{u(t')}\, dt' \geqq 0,$$

Then, as $k = -1, 0$ if $H_0 \leq 0$ we have $H < 0$ for every time.

Since $a(t) = a_0 \exp \int_0^t H(t')\, dt'$, we have

$$a(t) = a_0\, e^{H_0\ t}\ e^{k \int_0^t g(t')\ dt'}\ e^{-3\int_0^t f(t')\ dt'} \geqq 0.$$

Then if we want that the scale a would vanish (Felder *et al.*, 2002; Steinhardt and Turok, 2002a,b),[4] at a finite time t_1 it is necessary that either $\int_0^{t_1} f(t')\ dt'$ would be divergent or, when $k = -1, 0$, that $\int_0^{t_1} g(t')\ \ dt'$ or $\int_0^{t_1} f(t')\ dt'$ would be divergent.

[4] This is the condition for the big-crunch of for cyclic universes, considered here in a more general case than in Felder *et al.* (2002).

3. FIXED POINT

The fixed point in space $\{H^*, \psi_i^*, \overset{\bullet}{\psi}{}_i^*\}$, $i = 1, \ldots, n$, of the system are given by Eqs. (10) and (11), satisfying condition (12), and they exist only in the flat-space case $k = 0$. They satisfy

$$\overset{\bullet}{\psi}{}_i^* = 0, \quad i = 1, \ldots, n,$$
$$\partial_{\psi_i} V(\psi_1^*, \ldots, \psi_n^*) = 0, \quad i = 1, \ldots, n,$$

but, from the Einstein condition, we have that

$$H^{*\,2} = 2\; V(\psi_1^*, \ldots, \psi_n^*)$$

so there are fixed points only in the case $V^* = V(\psi_1^*, \ldots, \psi_n^*) \geq 0$ and $\partial_{\psi_i} V(\psi_1^*, \ldots, \psi_n^*) = 0$.

In these fixed points the energy ρ takes the value V^* and the pressure p takes the value $-\,V^* \leq 0$ and w takes the value -1. In the fixed point, the universe always is expansive and the evolution of the scale factor a is exponential when $V^* > 0$. Finally it is easy to demonstrate that the fixed points are saddle-points.

Let us give a simple example for the systems with fixed points the potential around these points must be in the simplest case a polynomial of degree two

$$V(\psi_1, \ldots, \psi_n) = V^* + \sum_{i=1}^{n} w_i\; (\psi_i - \psi_i^*)^2$$

where the coefficients w_i are such that the Hessian at the fixed points would be positive. The fluctuation around these points can be studied with the methods of papers (Castagnino *et al.*, 2002, 2003).

4. LYAPUNOV FUNCTION

In this section we introduce a Lyapunov function to demonstrate that the scale factor oscillations are only possible when $k = +1$, for any non-negative potential. Considering the potential V is non-negative definite, we can define a simple Lyapunov function as

$$\mathcal{L} = u \;= a^2, \tag{14}$$

which also is definite non-negative. Its derivative, a long a solution curve, can be found from Eq. (12), that we can write as

$$\overset{\bullet}{u} = \pm 2\,u \left(-\frac{k}{u} + 2\,V + \sum_{i=1}^{n} \overset{\bullet}{\psi}{}_i^{2} \right)^{1/2}. \tag{15}$$

Then we have the following cases

(i) $k = -1$. Let us first consider the case $\overset{\bullet}{u}(t_0) > 0$, then we must take the $+$ sign in the last equation, and $\overset{\bullet}{u}$ never vanishes so u is monotonous and therefore a Lyapunov function. On the contrary, if $\overset{\bullet}{u}(t_0) < 0$, we must take the sign $-$ and we have a contractive solution. Then $\overset{\bullet}{u}$ vanish only if $u = 0$ which corresponds to a singularity and therefore to the end of the evolution, thus u is also monotonous.

(ii) $k = 0$. Let us first consider the case $\overset{\bullet}{u}(t_0) > 0$, again we must take the $+$ sign. According to the discussion about the fixed point, $\overset{\bullet}{u}$ can only vanish asymptotically if $V^* = 0$, namely $\overset{\bullet}{u}$ does not vanish for finite times and therefore u is monotonous. In the other case, the evolution is contractive if, $\overset{\bullet}{u}(t_0) < 0$, and we must take the $-$ sign. This sign cannot change until a time where $u = 0$. There, according to what we have said about the fixed points, $\overset{\bullet}{u}$ vanishes asymptotically if $V^* = 0$, namely $\overset{\bullet}{u}$ does not vanish in finite times and therefore in both cases u is monotonous.

(iii) If $k = +1$, $\overset{\bullet}{u}$ can vanish and therefore we cannot say if it has a monotonous behavior. So only in this case u can correspond to an oscillatory solution.

5. SCALAR FIELD EVOLUTION

In this section we study the qualitative behavior of the scalar fields and the energy density, for an arbitrary potential. As the potential remains arbitrary we cannot show the detailed evolution but we can find its qualitative properties. Let us consider n field with potential $V(\psi_1, \ldots, \psi_n)$ and Eqs. (10), multiplied by $\overset{\bullet}{\psi}_i$ and added in indices $i = 1, \ldots, n$, namely

$$E = \rho\,(\psi_1, \ldots, \psi_n, \overset{\bullet}{\psi}_1, \ldots, \overset{\bullet}{\psi}_n) + \sum_{i=1}^{n} \int_0^t 3\,H\,\overset{\bullet}{\psi}{}_i^2\, dt'\,, \tag{16}$$

where $\rho = \frac{1}{2}\sum_{i=1}^{n} \overset{\bullet}{\psi}{}_i^2 + V$ is the energy density and E is a integration constant. Then, the variation of the energy density along a trajectory is

$$\overset{\bullet}{\rho} = -3\,H \sum_{i=1}^{n} \overset{\bullet}{\psi}{}_i^2, \tag{17}$$

so the sign of $\overset{\bullet}{\rho}$ is completely defined by the sign of H. Let us consider the particular case $V \geq 0$ and therefore from Eq. (8) the density is not negative. While the sign of the Hubble function would be non-negative, ρ would be decreasing as ψ_i^2, $i = 1, \ldots, n$, and the density ρ will converge to a constat ρ_∞ either zero or positive. As we show in Section 2 when $k = -1, 0$, the dynamic is relatively simple since $\dot{H} \leq 0$. When $k = 1$ the dynamic is more complex and it will be discussed it in the next section, so in the next paragraph we will only consider the problem of finding the value of ρ_∞ when $k = -1, 0$.

(i) If the $\rho_\infty = 0$, the potential $V \to 0$ and the kinetic terms $\dot{\psi}_i^2/2 \to 0$, and therefore $\psi_i \to cont.$, and thus $\ddot{\psi}_i \to 0$. From Klein Gordon equation we conclude that $\partial_{\psi_i} V \to 0$. Then the trajectories go to a minimum potential, which is zero and therefore also the pressure $p \to 0$ but w has no limit. From Eq. (12), we see that when $k = 0$, the trajectories go to a fixed point. When $k = -1$, using the properties of the Lyapunov function contained in Eq. (14), we see that $u \to \infty$ and $H \to 0$. Then, the oscillatory or monotonous behavior of the fields depends on the properties of the potential.

(ii) If $\rho_\infty > 0$, since ρ has a monotonous behavior, considering Eq. (17), we conclude that $H \to 0$ and/or $\sum_{i=1}^{n} \dot{\psi}_i^2 \to 0$. If then kinetic terms vanish, and therefore $\psi_i \to 0$. The fields evolve in such a way to make V a minimum, but since $\rho_\infty > 0$, this minimum is $V_m = \rho_\infty > 0$ and $w = \rho/p = -1$; so we have inflation. Then from Eqs. (11) and (12), we conclude that $H \to H_\infty > 0$ and $\dot{H} \to 0$.

(iii) Finally we have the case $\rho_\infty > 0$. If $H \to 0$. Then using the Eq. (12), we see that the only possibility is $k = +1$ since $V,\ u,\ \sum_{i=1}^{n} \dot{\psi}_i^2$ are positive. In Section 7 we will consider the particular $u = const.$

When H is always negative the universe evolution is a contraction. This case has being deeply studied in Felder *et al.* (2002), Steinhardt and Turok (2002a,b).

(iv) If $\rho \to \infty$ then $V \to \infty$ and / or $\sum_{i=1}^{n} \dot{\psi}_i^2 \to \infty$ but in both cases from Eq. (12) we conclude $H \to -\infty$, in which case the universe remains trapped in the singularity $a = 0$ at a finite time t_1 since $\dot{H} < 0$ and, from Eq. (6), $\ddot{a} < 0$.

(v) If $\rho \to \rho_\infty > 0$, as ρ goes monotonously to a finite constant, then $\dot{\rho} \to 0$. From Eq. (17), as $H < 0$ and decreasing then $\dot{\psi}_i^2 \to 0$, and therefore $\psi_i \to cont.$, so $\ddot{\psi}_i \to 0$ and using Klein Gordon equation we conclude that $\partial_{\psi_i} V \to 0,\ i = 1, \ldots, n$. When $k = 0$, from Eq. (11) we have $H \to -\alpha^2$, then u decreases exponentially to zero.

On the contrary if $k = -1$, from Eqs. (11), (17), $\dot{H} \simeq -1/u$. Then if u would tend to a non-vanishing constat we would have $H \to -\infty$, but this is

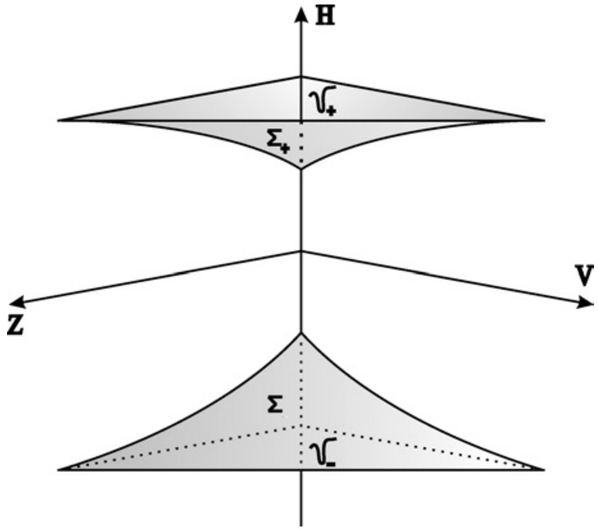

Fig. 1. $H,\ V,\ Z$ space for $V > 0$.

a contradiction since $\overset{\bullet}{u} = 0$ because u is constant. Then the only possible alternative is that $u \to 0$, in which case from Eq. (12), we do have $H \to -\infty$.

6. HVZ SPACE

Let us qualitatively describe the solutions as curves in a space with variables $\{H, V, Z\}$. In this section we consider that the only solutions with physical interest are those that $\overset{\bullet}{a}\ (t = 0) > 0$ namely those with an expansive initial behavior. Then we define a simple tridimensional space of variables $H,\ V$, and $Z = \frac{1}{2}\sum_{i=1}^{n} \overset{\bullet}{\psi}{}_i^2 = 0$ (see Figs. 1–3) where H varies in the interval $(-\infty,\ \infty)$, Z is non-negative and the potential V takes values in the interval $(V_{\min},\ \infty)$, where $V_{\min}$ is global minimum. We can rewrite Eq. (12) as

$$H^2 = H_0^2 - \frac{k}{u} \ , \tag{18}$$

where

$$H_0^2 = 2\,\rho \ , \tag{19}$$

and the density of energy is rewritten as

$$\rho = Z + V.$$

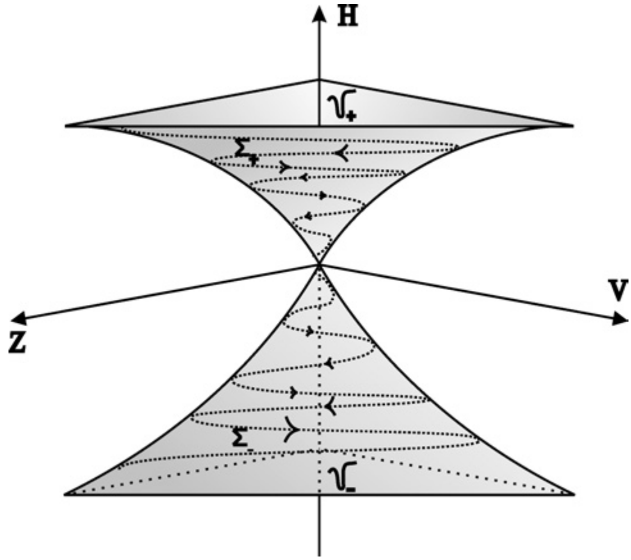

Fig. 2. $H,\ V,\ Z$ space for $V = 0$.

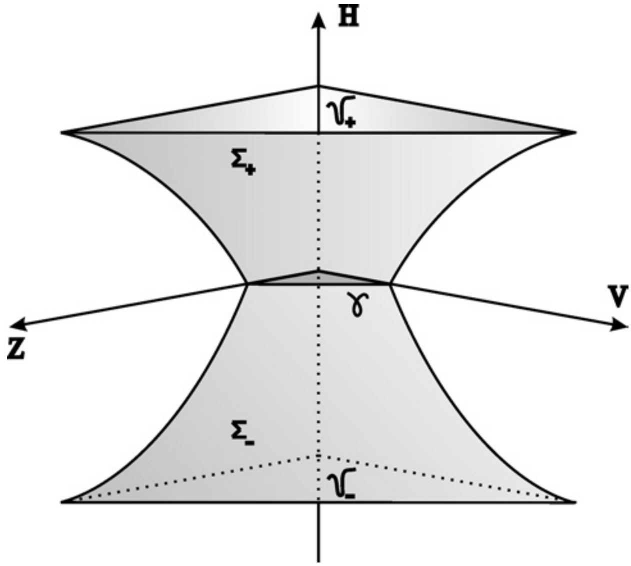

Fig. 3. $H,\ V,\ Z$ space for $V < 0$.

Then, using these variables $H,\ V,\ Z$ from the properties of fixed points we can deduce that, when $k = 0$, the only fixed point is $H^*_\pm = \pm\sqrt{2\ V^*},\ \ V^* \geq 0$ and $Z^* = 0$.

Using Eqs. (6) and (8) we have

$$\ddot{a} = 2\ a\ (V - 2Z) = 0.$$

From Eqs. (8), the plane V, Z has two relevant separatrices (not shown in the figures): one corresponds to $p = 0$ defined by $Z = V$, and therefore, in the region $V_{\min} < V < Z$, the pressure is always positive or it is negative if $V > Z$. The second separatrix is defined by $V = 2\ Z$ that corresponds to $\ddot{a} = 0$, then, if $V_{\min} < V < 2\,Z$ we have $\ddot{a} < 0$; and if $V > 2\,Z$ we have $\ddot{a} > 0$. We easily understand that in the intersection of these regions $\ddot{a} > 0$ implying that $p < 0$, for any potential since $V > 2\,Z > Z$.

The Eq. (19) defines two surfaces $\Sigma_\pm$ in space $H,\ Z,\ V$, namely

$$H_0 = \pm\sqrt{2\ \rho}\,.$$

When $V > 0$ (Fig. 1) the surfaces $\Sigma_\pm$ have neither mutual intersection nor with the plane $H = 0$. If the potential can take the value $V = 0$ (Fig. 2) then $\Sigma_\pm$ intersects in a vertex defined by $(H = 0,\ V = 0,\ Z = 0)$. When the potential can take negative values, $V < 0$ (Fig. 3) the surfaces intersect in a curve γ on the $H = 0$ plane.

According to the values of k we have the following cases:

(i) From Eq. (18) we can see that when $k = 0$ the solutions are contained either in Σ_+ or Σ_-.
(ii) Moroever from the same equation, when $k = -1$ the solutions are contained in the interior of surfaces Σ_+ and Σ_-.
(iii) And when $k = 1$ the solutions are contained in the exterior of surfaces Σ_+ and Σ_-.

Combining $V > 0,\ V = 0,\ V < 0$ with $k = -1,\ 0,\ 1$ we have nine different cases.

Cases 1 and 2—When $k = 0$, and $V \geqq 0$, (see Figs. 1 and 2), the solutions cannot go from Σ_+ to Σ_- since they do not intersect (see Fig. 1) or they intersect just in a fixed point (Fig. 2). From Eq. (11) and Eq. (17), the trajectories contained in Σ_+ evolve in the direction in which H decrease, ρ decrease and therefore Z and V asymptotically converge to the vertex defined by the fix points $(H^*,\ V^*,\ Z^*)$, but the universe can also be expansive, since we may have decreasing $H > 0$, then the scale factor has only two possibilities: either $u \to U > 0$ (a finite positive value of u) or $u \to \infty$. The solutions in Σ_-, using Eq. (11) and (17) evolve in the direction that H^2 and ρ grow, these correspond to contracting universes that will collapse in a finite time t_1, namely $\lim_{t\to t_1} a = 0$ and $\lim_{t\to t_1} H = -\infty$ (see Castagnino *et al.*, 2000, 2001, 2002; Felder *et al.*, 2002; Steinhardt and Turok, 2002a,b; Calzetta and El Hasi, 1993, 1995; Bombelli *et al.*, 1998, 1999). The case $V = 0$ is described in Fig. 2 where we have drawn on illustrative evolution corresponding to the oscillatory field case.

Case 3—Let us now consider the case when $k = 0$ and there exists a domain where $V < 0$, (see Fig. 3) then the trajectories begin in Σ_+ and we have a universe expansion. As H decreases the trajectory intersects the curve γ, then H changes sign and the trajectory approach Σ_-, where the evolution of the universe is a contraction that finally collapse as $a = 0$. Then *only* if the potential take negative values the universe can have an expansive phase followed by a contacting one and to collapse at a finite time (e.g. Felder *et al.*, 2002; Steinhardt and Turok, 2002a,b).

Case 4—When $k = -1$ and $V > 0$, (Fig. 1) from Eq. (18), we see that the expansive solutions are contained in a unbounded domain v_+ limited by the lower bound Σ_+ and the planes $Z = 0$ and $V = V_{\min}$, while the contracting trajectories belong to a volume v_- with the upper bound Σ_-. In both cases the trajectories are confined either in the upper region v_+ ($H > 0$) or in the lower region v_- ($H < 0$), since Σ_+, Σ_- do no intersect each other. The solutions contained in v_+, from Eq. (11), asymptotically converge to surface Σ_+ and since $\dot{H} < 0$, the energy density decrease and therefore Z and V also decrease, in such a way that necessarily the trajectory end in the fixed point. Instead, for the trajectories of v_- H^2 increases as well the energy density, V, and Z. The trajectories converge to Σ_- as $\lim_{t\to t_1} a = 0$. Even if the trajectories follows the sense of the decreasing of H, a trajectory cannot go from v_+ to v_- since the regions and v_- and v_+ do not intersect.

Case 5—The case $k = -1$and $V = 0$ (see Fig. 2) is the limit case of the preceding one.

Case 6—When $k = -1$ and there is a domain where $V < 0$, the universe may have an expansion period and then collapse, which correspond to go from volume v_+ to volume v_-, the transit from one region to the other take place in the plane $H = 0$ in a domain defined by the curve γ, and the segments $Z = 0$ and $V = V_{\min}$. This surface contains all the points where the evolution changes from expansive to contractive.

Cases 7, 8, and 9—When $k = +1$, the sign of H is not fix and the trajectories belong to the exterior domain limited by $\Sigma_\pm$. Then we can have expanding, collapsing, or oscillatory solutions for the scale factor. It is important to remark that with this geometry it is not necessary that the potential would be negative for the existence of cyclic universes. In our previous work on Brane dynamic (Lara and Castagnino, 2004) the oscillatory solution corresponds to $k = 0$ and $C < 0$, since the result of a spheric geometry can be reproduced adding dark radiation in the usual spatially flat model.

7. PARTICULAR SOLUTIONS

7.1. Oscillatory Solution for the Scalar Fields

Let us consider a potential containing quartic terms for each field and also quartic interaction terms (Castagnino *et al.*, 2000, 2001, 2002; Cornish and Lewin, 1996), precisely

$$V = V_0 + \sum_{i=1}^{n} \left(\frac{m_i^2}{2} \psi_i^2 + \frac{\Omega_i}{4} \psi_i^4 \right) + \sum_{i,\, j=1; i \neq j}^{n} \frac{\lambda_{ij}^2}{2} \psi_i^2 \psi_j^2 , \tag{20}$$

where V_0 is the cosmological constant.

To simplify our analysis let us only consider two coupled fields. Let us consider a choice of the parameter such that a period of time I where $H \simeq 0$ would exit or better considering Eq. (10)

$$|3H \overset{\bullet}{\psi}_1 | \ll | \overset{\bullet\bullet}{\psi}_1 |, \quad \left| \left(m_1^2 + \lambda \psi_2^2 + \Omega_1 \psi_1^2 \right) \psi_1 \right|,$$

$$|3H \overset{\bullet}{\psi}_2 | \ll | \overset{\bullet\bullet}{\psi}_2 |, \quad \left| \left(m_2^2 + \lambda \psi_1^2 + \Omega_2 \psi_2^2 \right) \psi_2 \right|.$$

Then Eqs. (10) can be approximated as

$$\overset{\bullet\bullet}{\psi}_1 + \left(m_1^2 + \lambda \psi_2^2 + \Omega_1 \psi_1^2 \right) \psi_1 = 0, \tag{21}$$

$$\overset{\bullet\bullet}{\psi}_2 + \left(m_2^2 + \lambda \psi_1^2 + \Omega_2 \psi_2^2 \right) \psi_2 = 0, \tag{22}$$

namely the equation studied by Chiricov (1979), Chirikov and Shepelyansky (1981). Then in period I the behavior of the scalar fields may be chaotic, precisely transitorily chaotic, since it only correspond to period I (for a numerical see Easther and Maeda, 1999; Felder *et al.*, 2002). This transitory chaotic behavior can happen either in the transition between the expansive and contractive phases in the geometries $k = -1, 0, 1$ or as the transition between two inflationary phases in the spheric geometry $k = 1$, as follows from the equations of the previous section. As a numerical example it is easy to verify the existence of this kind of behavior in the models with the following parameters: $m_{1,2} = 1$, $\Omega_{1,2} = 0$, $\lambda = 30$, $\psi_1(0) = 0$, $\overset{\bullet}{\psi}_1 (0) = 1$, $\psi_2(0) = 1$, $\overset{\bullet}{\psi}_2 (0) = 0$. This transitory chaos in the fields and therefore in the density and the pressure, could be useful to explain the homogeneity of the spacial universe avoiding unnecessary fine tunnings in the initial conditions.

7.2. Periodic Solutions for the Scale Factor

In Sections 5 and 6 we have shown the possible existence of periodic solutions when $k = +1$. As an example let us consider a FRW with just one scalar field,

$k = +1$ and the potential $V = V_0 + 1/2\ \psi^2$ where for the sake of simplicity and the scale invariance of the equations we have taken the field mass $m = 1$. Using Eqs. (10), (11), and (12) we have

$$\ddot{\psi} + 3H\dot{\psi} + \psi = 0, \tag{23}$$

$$\dot{H} = \frac{1}{u} - 3\dot{\psi}^2, \tag{24}$$

$$0 = -H^2 - \frac{1}{u} + 2V_0 + \psi^2 + \dot{\psi}^2 . \tag{25}$$

To find a periodic solution we make the ansatz

$$|3H\dot{\psi}| \ll |\ddot{\psi}|, |\psi|, \tag{26}$$

Integrating Eq. (23) we obtain a particular solution $\psi = \psi_0 \sin t$. Now considering a very small oscillation amplitude with respect to u, u_0 we obtain

$$\begin{aligned} u &= u_0 + \varepsilon f(t), \\ |\varepsilon f(t)| &\ll u_0. \end{aligned} \tag{27}$$

then Eq. (24) reads

$$\dot{H} = \frac{k}{u_0} - 3\dot{\psi}_0^2 \sin^2 t .$$

Integrating we obtain

$$H = C_1 + \frac{t}{u_0} - \frac{3}{2}\psi_0^2\ t + \frac{3}{4}\psi_0^2\ \sin 2t,$$

where C_1 is the integration constant that we will take equal to zero. Since we are trying to find oscillatory solution let us take the initial condition satisfying the equation $\psi_0^2 = \frac{2}{3} u_0^{-1}$, then

$$H \cong \frac{3}{4}\psi_0^2 \sin 2t, \tag{28}$$

Putting all these approximative results in Eq. (25) we obtain

$$V_0 = \frac{1}{6}\frac{1}{u_0}.$$

Now it is easy to show that these approximation satisfy the ansatz of Eq. (26). Then in this case V_0 is completely determined and also the particular oscillatory solutions for H and ψ.

Finally, using Eqs. (27) and (28) it turns out that $\varepsilon\, f(t) = -\cos 2\,t + \alpha$ where α is an arbitrary integration constant that we can take it equal to zero, since u_0 we can change by $u_0 - \alpha$.

7.3. Solution with Stationary Energy Density

In the case of closed universe $k = +1$, with n scalar field an a generic potential $V(\psi_1, ..., \psi_n)$, we will see if it is possible that the energy density, of the kinetic terms, would remain approximately constant along a solution curve. Then we call

$$C = 3 \sum_{i=1}^{n} \dot{\psi}_i^2 \cong const. \tag{29}$$

Using Eq. (11) and the definition of the Hubble function, we find the system

$$\begin{aligned} \dot{H} &\cong \frac{1}{u} - C, \\ \dot{u} &= 2\,u\,H\,, \end{aligned} \tag{30}$$

with fix point $H^* = 0,\ \ u^* = 1/C$. Via a simple study of the vector field of Eqs. (30) we can see that this fixed point is a center, such that in its neighborhood oscillatory solution for the Hubble function may exist as well as for the scalar factor $a(t)$.

As a example, let us consider a very simple case where there is exact conservation of the kinetic energy. Let us also consider two scalar fields ψ_1, ψ_2 and the potential $V = V_0 + m^2 / 2\ (\psi_1^2 + \psi_2^2)$ and initial condition such that $u = u^*$. Using these conditions and integrating the Klein–Gordon equation for both fields we obtain

$$\psi_{1,2} = \alpha_{1,2}\, \sin m\,t + \beta_{1,2}\, \cos m\,t$$

where $\alpha_{1,\,2}$ and $\beta_{1,\,2}$ are integration constants. Taking into account the fields of Eqs. (11) and (12) we obtain

$$\begin{aligned} \alpha_1 &= \beta_2 = \pm\, \frac{\sqrt{2\,V_0}}{m}\,, \\ \alpha_2 &= \beta_1 = \,0\,, \end{aligned}$$

and the stationary solution $u^* = (6\,V_0)^{-1}$, where the kinetic energy density is $V_0 / 2$, the energy density is $\rho = 2\,V_0$ and the pressure $p = -\,V_0$. It is interesting to observe that the obtained solution is a limit cycle, namely an isolated periodic solution for each component of the field when the Hubble function vanishes.

8. CONCLUSION

In this work we have find and developed a set of properties of the cosmological FRW models coupled with n minimally coupled scalar fields and an arbitrary potential. We have rigorously proved that the plane and hyperbolic geometries only have the following evolution for the scale factor:

(i) Expansive.
(ii) Collapsing in a monotonous way.
(iii) A expansive phase followed by a contractive one and then a collapse that may eventually generate a cyclic universe (Steinhardt and Turok, 2002a,b) which is only possible if the potential takes, in a finite domain, negative values.

The oscillatory behavior of the scale factor appears only in the spheric geometry $k = +1$. In this geometry expansion and contraction of the scale factor are also possible, and these alternatives depend on the initial conditions.

We hope that these results would be useful to obtain a better understanding of the properties of the models endowed with different kinds of potentials.

REFERENCES

Bombelli, L., Castagnino, M., and Lombardo, F. (1998). *Journal of Mathematical Physics* **39**, 6040.
Bombelli, L., Castagnino, M., and Lombardo, F. (1999). *Modern Physics Letters* **14**, 537.
Calzetta, E. and El Hasi, C. (1993). *Classical Quantum Gravity* **10**, 1825.
Calzetta, E. and El Hasi, C. (1995). *Physical Review D* **51**, 2713.
Castagnino, M. Giacomini, H., and Lara, L. (2000). *Physical Review D* **61**, 107302.
Castagnino, M., Giacomini, H., and Lara, L. (2001). *Physical Review D* **63**, 044003.
Castagnino, M., Giacomini, H., and Lara, L. (2002). *Physical Review D* **65**, 023509.
Castagnino, M., Chavarriga, J., and Lara, L. (2003). *General Rel Grav* **35**, 8.
Castagnino, M., Chavarriga, J., Lara, L., and Grau, M. (2002). *International Journal of Theoretical Physics* **41**(11), 2027–2035.
Charters, T. (2001). *Classical Quantum Gravity* **18**, 1703.
Chiricov, B. (1979). *Physical Report* **52**, 265.
Chirikov, B. and Shepelyansky, D. (1981). *JETP Letters* **34**, 163.
Cornish, N. and Lewin, J. (1996). *Physical Review D* **53**, 3022.
Easther, R. and Maeda, K. (1999). *Classical Quantum Gravity* **16**, 1637.
Felder, G., Frolov, A., Kofman, L., and Linde, A. (2002). *Physical Review D* **66**, 023507.
Jackson, E. (1991). *Perspectives of Nonlinear Dynamics*, Cambridge University Press, Cambridge, England.
Kolda, C. and Lyth, D. H. (1999c). *Physical Letters B* **458**, 197.
Lara, L. and Castagnino, M. (2004). *Physical Review D* **70**, 043535.
Liddle, A. R. and Scherrer, R. J. (1999b). *Physical Review D* **59**, 023509.
Linde, A. (1983). *Physical Letters B* **129**, 177.
Mizuno, S., Maeda, K.-C., Yamamoto, J. (2003). *Physical Review D* **67**, 023516.
Peebles, P. (1993). *Principles of Physical Cosmology*, Princeton University Press, Princeton, NJ.
Peebles, P. J. E. and Ratra, B. (1988). *Astrophysics Journal* **325**, L17.

Peebles, P. J. E. and Vilenkin, A. (1999a). *Physical Review D* **59**, 063505.
Ratra, B. and Peebles, P. J. E. (1988). *Physical Review D* **37**, 3406.
Steinhardt, P. J. and Turok, N. (2002a). *Science* **296**, 1436.
Steinhardt, P. J. and Turok, N. (2002b). *Physical Review D* **65**, 126003.
Strogatz, S. (1994). *Nonlinear Dynamics and Chaos*, Addison-Wesley, New York.
Tsujikawa, S., *et al*. (2000). xxx.lanl.gov, hep-ph/0012141.